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Generating matrices of the k-Jacobsthal Numbers

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Abstract: In this paper we define some tridiagonal matrices depending of a parameter from which we will find the k-jacobsthal numbers. And from the cofactor matrix of one of these matrices we will prove some formulas for the k-jacobsthal numbers differently to the traditional form. Finally we will study the eigenvalues of these tridiagonal matrices.

1. INTRODUCTION

The generalization of the Fibonacci has been treated by some authors eg. Hoggat V.E and Horadom A.F. k- Fibonacci generalizations has been found by Falcon .S and plaza A to study the method of triangulation.

In this paper we have to give generating matrices of the k -jacobsthal numbers Besides the usual jacobsthal numbers many kinds of generalization of these have been presented and well known jacobsthal sequence is defined as $j_0 = 0, j_1 = 1, j_n = j_{n-1} + 2\, j_{n-2}$, for $n \geq 2$where j_n denotes the n-jacobsthal numbers for any positive real number k, the k-jacobsthal sequence say $\{ j_{k,n}\}_{n=0}^{\infty}$ is defined recurrently by

$j_{k,n+1} = k\, j_{k,n} + 2\, j_{k,n-1}$ with initial conditions as$j_{k,0} = 0,\ j_{k,1} = 1$

	$j_{k,n}$
0	0
1	1
2	k
3	$k^2 + 2$
4	$k^3 + 4k$
5	$k^4 + 6k^2 + 4$
6	$k^5 + 8k^3 + 12k$
7	$k^6 + 10k^4 + 24k^2 + 8$
8	$k^7 + 12k^5 + 40k^3 + 32k$
9	$k^8 + 14\, k^6 + 60k^4 + 80k^2 + 16$
10	$k^9 + 16\, k^7 + 84k^5 + 16k^3 + 80k$

Particular cases of the previous definitions are if k=1, the classical jacobsthal sequence obtained $j_0 = 0, j_1 = 1, j_{n+1} = j_n + 2\, j_{n-1}$ for $n \geq 1$, $\{j_{k,n}\}_{n=0}^{\infty}$ = {0,1,1,3,5,11….}

2. Tridiagonal matrices and k-jacobsthal numbers

In this section we extend the matrices defined and applied them to the k-jacobsthal numbers in order to prove some formulas differently to the traditional form.

2.1 The determinant of a special kind of the tridiagonal matrices

Let us consider the n by n tridiagonal matrices

$$M_n = \begin{pmatrix} a & b & & & & \\ c & d & e & & & \\ & c & d & e & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & d & e \\ & & & & c & d \end{pmatrix}$$

Solving the sequence of determinants, we find

$$|M_1| = a$$
$$|M_2| = d.|M_1| - bc$$
$$|M_3| = d.|M_2| - ce.|M_1|$$
$$|M_4| = d.|M_3| - ce.|M_2|$$
$$\ldots.$$

In general

$$|M_{n+1}| = d.|M_n| - ce.|M_{n-1}|$$

2.2 Some tridiagonal matrices and k-jacobsthal numbers

$If\ a = d = k, b = e = 2\ and\ c = -1$, the matrices Mn transformed in the tridiagonal matrices

$$H_n(k) = \begin{pmatrix} k & 2 & & & & \\ -1 & k & 2 & & & \\ & -1 & k & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & k & 2 \\ & & & & -1 & k \end{pmatrix}$$

In this case and taking

$|H_1(k)| = k = j_{k,2}$
$|H_2(k)| = k.k - 2(-1) = k^2 + 2 = j_{k,3}$
$|H_3(k)| = k.(k^2 + 2) - 2(-1).k = k^3 + 4k = j_{k,4}$

And formula (2.1) is $|H_1(k)| = j_{k,n+1}\, for\ n \geq 1$
The k-jacobsthal numbers can also be obtained from the symmetric tridiagonal matrices

$$H'_n(k) = \begin{pmatrix} k & i & & & & \\ i & k & i & & & \\ & i & k & i & & \\ & & \ddots & \ddots & \ddots & \\ & & & i & k & i \\ & & & & i & k \end{pmatrix}$$

Where i is the imaginary unit,i.e $i^2 = -1$

$* \, if \; a = k^2 + 2 \;\; d = k^2 + 4, \;\; b = e = c = 2$, the tridiagonal matrices are

$$O_n(k) = \begin{pmatrix} k^2+2 & 2 & & & & \\ 2 & k^2+4 & 2 & & & \\ & 2 & k^2+4 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & k^2+4 & 2 \\ & & & & 2 & k^2+4 \end{pmatrix}$$

In this case , it is
$|O_n(k)| = j_{k,2n+1} \; for \; n \geq 1$ So, with $|O_0(k)| = j_{k,1} = 1$, the sequence of these determinants is the sequence of odd $k - jacobsthal \; numbers \, \{1, k^2+2, k^4+6k^2+4, k^6+10k^4+24k^2+4 \ldots.\}$
*Finally $If \; a = k \;\; d = k^2 + 4, \;\; b = 0, e = c = 2 \; for \; n \geq 1$

$$|E_n(k)| = \begin{pmatrix} k & 0 & & & & \\ 2 & k^2+4 & 2 & & & \\ & 2 & k^2+4 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & k^2+4 & 2 \\ & & & & 2 & k^2+4 \end{pmatrix}$$

$because \; |E_n(k)| = j_{k,2n} \; for \; n \geq 1$ So, with $|E_0(k)| = j_{k,0} = 0$

Cofactor matrices of the generating matrices of the k-jacobsthal numbers

The following definitions are well Known:

If A is a square matrix, then the minor of its entry a_{ij} , also known as the (i,j) minor of A, is denoted by M_{ij} and is defined to be the determinant of the submatrix obtained by removing from A its $i - th \; row \; and \; j - th \; column$.

If follows $C_{ij} = (-1)^{i+j} M_{ij}$ and C_{ij} called the cofactor of a_{ij} , also refered to as the (i,j) cofactor of A.

Define the cofactor matrix of A,as the $n \, X \, n \; matrix \; C$ whose (i,j) entry is the (i,j) cofactor of A.

Finally, the inverse matrix of A is $A^{-1} = \frac{1}{|A|} C^T$, where $|A|$ is the determinant of the matrix A (assuming non zero) and C^T is the transpose of the cofactor matrix C or adjugate matrix of A.

On the other hand, let us consider the $n \, X \, n$ nonsingular tridiagonal matrix

$$T = \begin{pmatrix} a_1 & b_1 & & & & & \\ c_1 & a_2 & b_2 & & & & \\ & c_2 & a_3 & b_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & & & c_{n-1} & a_n \end{pmatrix}$$

In an elegant and coincise formula for the inverse of the tridiagonal matrix $T^{-1} = (t_{i,j})$:

$$t_{ij} = \begin{cases} (-1)^{i+j} \frac{1}{\theta_n} b_i \ldots \ldots . b_{j-1} \theta_{i-1} \varphi_{j+1} & if\ \ i \le j \\ (-1)^{i+j} \frac{1}{\theta_n} c_j \ldots \ldots . c_{i-1} \theta_{j-1} \varphi_{i+1} & if\ \ i > j \end{cases}$$

where

θ_i verify the recurrence relation $\theta_i = a_i \theta_{i-1} - b_{i-1}\, \theta_{i-2}\ for\ i = 2, \ldots \ldots n$ with the initial conditions $\theta_0 = 1\ and\ \theta_1 = a_1$. Type equation here.

Formula (2.1) is one special case of this one.

φ_i verify the recurrence relation $\varphi_i = a_i \theta_{i+1} - b_i c_i\ \varphi_{i+2}\ for\ i = n-1, \ldots \ldots 1$ with the initial conditions $\varphi_{n+1} = 1\ and\ \varphi_n = a_n$ .observe that $\varphi_n = \det(T)$.

3.1 Cofactor matrix of $H_n(k)$

For the matrix $H_n(k)$, it is $a_i = k, b_i = 2, c_i = -1\ \theta_i = j_{k,i+1}$ and $\theta_i = j_{k,n-j+2}$

Consequently

$$((H_n(k))^{-1})_{i,j} = \begin{cases} (-1)^{i+j} \frac{1}{j_{k,n+1}} j_{k,i} \cdot j_{k,n-j+1} & if\ \ i \le j \\ \frac{1}{j_{k,n+1}} j_{k,j} \cdot j_{k,n-i+1} & if\ \ i > j \end{cases}$$

We will work with the cofactor matrix whose entries are

$$c_{i,j}(H_n(k)) = \begin{cases} (-1)^{i+j} j_{k,j} \cdot j_{k,n-i+1} & if\ \ i \ge j \\ j_{k,i} \cdot j_{k,n-j+1} & if\ \ i < j \end{cases}$$

So $c_{j,i}(H_n(k)) = (-1)^{i+j} c_{i,j}(H_n(k))$ if $i > j$

In this form, the cofactor matrix of $H_n(k)$ for $n \ge 2$ is

$$C_{n-1}(k) = \begin{pmatrix} j_{k,n} & 2j_{k,n-1} & j_{k,n-2} & j_{k,n-3} & \cdots & j_{k,2} & j_{k,1} \\ -j_{k,n-1} & j_{k,2}\, j_{k,n-1} & 2j_{k,2}\, j_{k,n-2} & j_{k,2}\, j_{k,n-3} & \cdots & j_{k,2}\, j_{k,2} & j_{k,2} \\ j_{k,n-2} & -j_{k,2}\, j_{k,n-2} & j_{k,3}\, j_{k,n-2} & 2j_{k,3}\, j_{k,n-3} & \cdots & j_{k,3}\, j_{k,2} & j_{k,3} \\ -j_{k,n-3} & j_{k,2}\, j_{k,n-3} & -j_{k,3}\, j_{k,n-3} & j_{k,4}\, j_{k,n-3} & \cdots & j_{k,4}\, j_{k,2} & j_{k,4} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ j_{k,2} & -j_{k,2}\, j_{k,2} & j_{k,3}\, j_{k,2} & -j_{k,4}\, j_{k,2} & \cdots & j_{k,n-1}\, j_{k,2} & 2j_{k,n-1} \\ -j_{k,1} & j_{k,2} & -j_{k,3} & j_{k,4} & \cdots & -j_{k,n-1} & j_{k,n} \end{pmatrix}$$

On the other hand taking into account of the inverse matrix $A^{-1} = \frac{1}{|A|} Adj(A)$,

It is easy to prove $|\,Adj(A)| = A^{n-1}$

So $|C_{n-1}| = j_{k,n+1}^{n-1}$
In this form, for $n = 2,3,4$ .... It is

$$C_1(k) = \begin{vmatrix} j_{k,2} & 2j_{k,1} \\ -j_{k,1} & j_{k,2} \end{vmatrix} = j_{k,3} \rightarrow j_{k,2}^2 + 2j_{k,1}^2 = j_{k,3}$$

$$C_2(k) = \begin{vmatrix} j_{k,3} & 2j_{k,2} & j_{k,1} \\ -j_{k,2} & j_{k,2}j_{k,2} & 2j_{k,2} \\ j_{k,1} & -j_{k,2} & j_{k,3} \end{vmatrix} = j_{k,4}^2$$

$\rightarrow \quad j_{k,2}^2(j_{k,3}^2 + 4j_{k,3}^2 + 4) = j_{k,4}^2$

$\rightarrow \left(\frac{j_{k,3-2j_{k,1}}}{k}\right)^2 \ \left(j_{k,3} + 2j_{k,1}\right)^2 = j_{k,4}^2$

$\rightarrow j_{k,3}^2 - 2j_{k,1}^2 = kj_{k,4}$

$$C_3(k) = \begin{vmatrix} j_{k,4} & 2j_{k,3} & j_{k,2} & j_{k,1} \\ -j_{k,3} & j_{k,3}j_{k,2} & 2j_{k,2}j_{k,2} & j_{k,2} \\ j_{k,2} & -j_{k,2}j_{k,2} & j_{k,3}j_{k,2} & 2j_{k,3} \\ -j_{k,1} & j_{k,2} & -j_{k,3} & j_{k,4} \end{vmatrix} = j_{k,5}^3 \quad \rightarrow \quad j_{k,5}^2\left(j_{k,3}^2 + 2j_{k,2}^2\right) = j_{k,5}^3$$

$\rightarrow (j_{k,3}^2 + 2j_{k,2}^2) = j_{k,5}$

$C_4(k) = j_{k,6}^4 \rightarrow \quad j_{k,3}^2(j_{k,4}^2 + 4j_{k,2}^2)j_{k,6}^2 = j_{k,6}^4$

$\rightarrow \left(\frac{j_{k,4-2j_{k,2}}}{k}\right)^2 \ \left(j_{k,4-2j_{k,2}}\right)^2 = j_{k,6}^2$

$\rightarrow j_{k,4}^2 - 2j_{k,2}^2 = kj_{k,6}$

Generating these results and taking into account $j_{k,n} = \frac{j_{k,n+1}-2j_{k,n-1}}{k}$, we find the following two formulas for k-jacobsthal numbers according to that n is odd or even $j_{k,n+1}^2 + 2j_{k,n}^2 = j_{k,2n+1}$ and $j_{k,n+1}^2 - 2j_{k,n-1}^2 = kj_{k,2n}$.

3.2 Cofactor matrix of $O_n(k)$

To apply (3.2) to the matrices $O_n(k)$, we must take into account that

$a_1 = k^2 + 2$
$a_i = k^2 + 4 \quad i \geq 2$
$b_i = c_i = 2 \quad i \geq 1$
$\theta_i = j_{k,2i-1}, \ i \geq 1$
$\varphi_j = \frac{1}{k} j_{k,2(n-i+2),} \ \ j \geq 1$ and consequently the cofactor of the (i, j) entry of these matrices is

$c_{i,j}(O_n(k)) = (-1)^{i+j} \frac{1}{k} j_{k,2j-1} j_{k,2(n-i+1),} \ \ i \geq j$

$c_{j,i}(O_n(k)) = c_{i,j}(O_n(k))$ for $j \geq i$

3.3 Cofactor matrix of $E_n(k)$

For the matrices $E_n(k)$, we must take into account that

$a_1 = k = j_{k,2}$
$a_i = k^2 + 4 \quad i \geq 2$
$b_1 = 0$
$b_{i+1} = c_i = 2 \quad i \geq 1$

$\theta_i = j_{k,2(i+1)},\ i \geq 1$
$\varphi_j = \frac{1}{k} j_{k,2(n-i+2)},\ j \geq 1$ and consequently the cofactor of the (i,j) entry of these matrices is
$c_{1,j}(E_n(k)) = (-1)^{i+j} \frac{1}{k} j_{k,2(n-i+1)},$

$c_{i,j}(E_n(k)) = (-1)^{i+j} \frac{1}{k} j_{k,2j} j_{k,2(n-i+1)},\ i \geq j$
$c_{j,i}(E_n(k)) = c_{i,j}(E_n(k))$ for $j > i > 1$

4. Eigen Values

This section is dedicated to the study of the eigen values of the matrices $H_n(k),\ O_n(k)\ and\ E_n(k)$.

4.1 Eigen Values of the matrices $\boldsymbol{H_n(k)}$

The matrix has entries in the diagonals $a_1, a_2, \ldots . a_n,$, $b_1, b_2, \ldots . b_{n-1},\ c_1, c_2, \ldots . c_{n-1}$
It is well known the eigen values of the matrix are
$\lambda_r = a + 2\sqrt{bc}\, cos\left(\frac{r\pi}{n+1}\right)$ for r =1,2,3,…,n.
Consequently, the eigen values of the matrix $H_n(k)$ where a = k, b = 2,c = -1 are $\lambda_r = k + 2i\sqrt{2}\, cos\left(\frac{r\pi}{n+1}\right)$.
If n is odd, then the matrix $H_n(k)$ has one unique real eigenvalue corresponding to r = $\frac{n+1}{2}$.
If n is even, no one eigen value is real.
So, the sequence of the tridiagonal $H_n(k)$ for n = 1,2,… is

$$\sum\nolimits_1 = \{k\}$$
$$\sum\nolimits_2 = \{k - i\sqrt{2}\}$$
$$\sum\nolimits_3 = \{k\ , k - 2i\}$$
…………..
……………

It is verified that $\sum_{j=1}^{n} \lambda_j = nk\ and\ \prod_{j=1}^{n} \lambda_j = j_{k,n+1}$ where $j_{k,n+1} = \prod_{j=1}^{n}\left(k + 2i\sqrt{2}\, cos\frac{\pi j}{n+1}\right)$.

Eigen values of the Matrices $\boldsymbol{O_n(k)}$

Matrices $O_n(k)$ are symmetric, so all its eigenvalues are real.

Theorem:

If λ_i is an eigenvalue of the matrix $O_n(k)$ for a fixed value k, then $\lambda_i + 2k + 1$ is eigenvalue of the matrix $O_n(k+1)$.

Proof:

If λ_i is an eigenvalue of the matrix $O_n(k)$, then it is

$$|O_n(k)\ - \lambda_i\, I_n| = \begin{bmatrix} k^2 + 2 - \lambda_i & 2 & \\ 2 & k^2 + 4 - \lambda_i & 2 \\ & 2 & k^2 + 4 - \lambda_i \end{bmatrix}$$

$$= \begin{bmatrix} (k+1)^2 - (\lambda_i + 2k + 1) & 2 & \\ 2 & (k+1)^2 + 2 - (\lambda_i + 2k + 1) & 2 \\ & 2 & (k+1)^2 + 2 - \lambda_i + 2k + 1 \end{bmatrix}$$

$$= |O_n(k+1)\ - (\lambda_i + 2k + 1)\, I_n|$$

Consequently, only it is necessary to find the eigenvalues of the matrix $O_n(1)$ for n = 2,3,… and then,

if λ_j is an eigenvalue of $O_n(1)$, then $\lambda'_j = \lambda_j + k^2 - 1$ is an eigenvalue of the matrix $O_n(k)$.

Eigen values of the matrices $E_n(k)$

Finally, we say a matrix is positive if all entries are real and non negative. If a matrix is tridiagonal and positive, then all the eigen values are real. So taking into account matrix $E_n(k)$ is tridiagonal and positive, all its eigenvalues are real.

Following the same process that for the matrices $O_n(k)$, we can prove that the first eigen value is k and othes verify $\lambda_i = \lambda_i(1) + k^2 - 1$.

Moreover $\sum_{j=1}^{n} \lambda_j\,(k) = (n-1)(k^2+4) + k$ and $\prod_{j=1}^{n} \lambda_j\,(k) = j_{k,2n}$

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